Voronoi diagrams and Bolza surface

Mikhail Bogdanov   Monique Teillaud

WoCG'14 - Kyoto

Outline

Outline

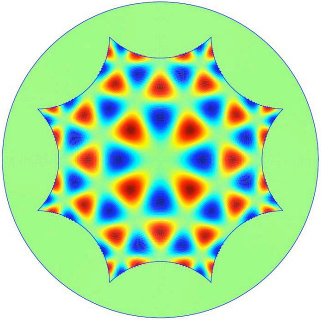
Applications

Examples

Physics

[Sausset, Tarjus, Viot]

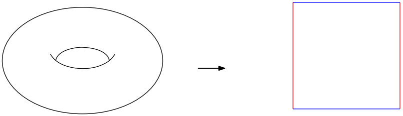
Neuromathematics



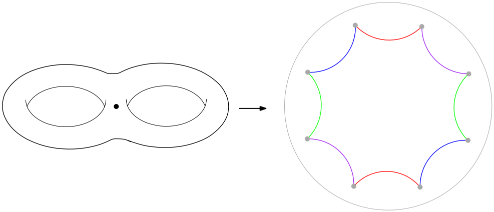
[Chossat, Faye, Faugeras]

Theory

From 1 handle to 2 handles



Flat torus
locally Euclidean metric



Bolza surface
Locally hyperbolic metric

Delaunay triangulations

- ∃ algorithms for the *d*D flat torus

  2d [Mazón, Recio], 3d [Dolbilin, Huson], *d*D [Caroli, T.]

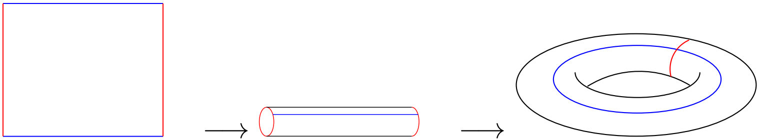
  ∃ software CGAL 2d [Kruithof], 3d [Caroli, T.]

- ∄ for the Bolza surface

Outline

The Flat Torus

locally Euclidean metric



$$G = \langle t_x, t_y \rangle$$

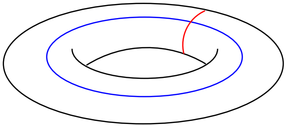
$$\mathbb{T}^2 = \mathbb{R}^2 / G$$

The Flat Torus

$G =< t_x, t_y >$

$\mathbb{T}^2 = \mathbb{R}^2 / G$

$\pi : \mathbb{R}^2 \to \mathbb{T}^2$



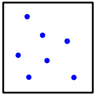
The Flat Torus

$\mathcal{P}$ finite point set

$G =< t_x, t_y >$

$\mathbb{T}^2 = \mathbb{R}^2 / G$

$\pi : \mathbb{R}^2 \to \mathbb{T}^2$



The Flat Torus

$G\mathcal{P}$ infinite point set

$G = < t_x, t_y >$

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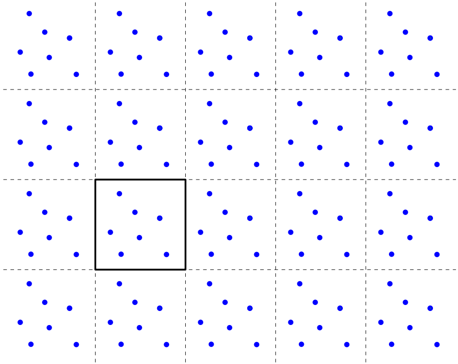


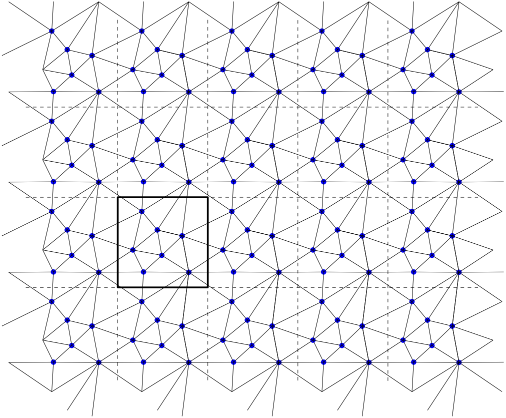
The Flat Torus

$G = < t_x, t_y >$

$\mathbb{T}^2 = \mathbb{R}^2 / G$

$\pi : \mathbb{R}^2 \to \mathbb{T}^2$

$DT(G\mathcal{P})$



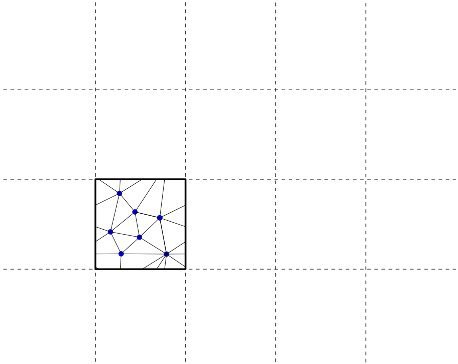
The Flat Torus

$$DT_{\mathbb{T}}(\pi(\mathcal{P})) = \pi(DT(G\mathcal{P}))$$ *(under some conditions)*

$G =< t_x, t_y >$

$\mathbb{T}^2 = \mathbb{R}^2/G$

$\pi : \mathbb{R}^2 \to \mathbb{T}^2$



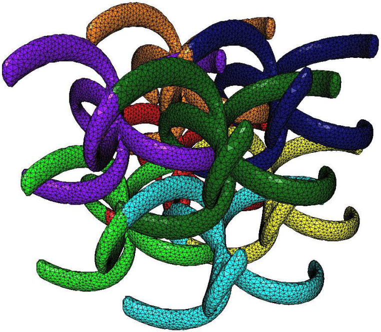
[Caroli, T.] *ESA'09, SoCG'11,* CGAL

The Flat Torus

$$G = < t_x, t_y >$$

$$\mathbb{T}^2 = \mathbb{R}^2 / G$$

$$\pi : \mathbb{R}^2 \to \mathbb{T}^2$$

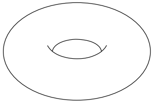


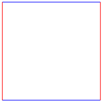
CGAL 3D periodic meshes (in progress)

From 1 to 2 handles

Flat torus

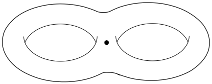
locally Euclidean metric

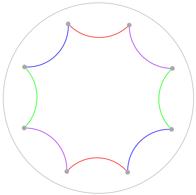




$$\mathbb{T}^2 = \mathbb{R}^2/G$$

$$G = < t_x, t_y >$$





$$\mathcal{M} = \mathbb{H}^2/\mathcal{G}$$

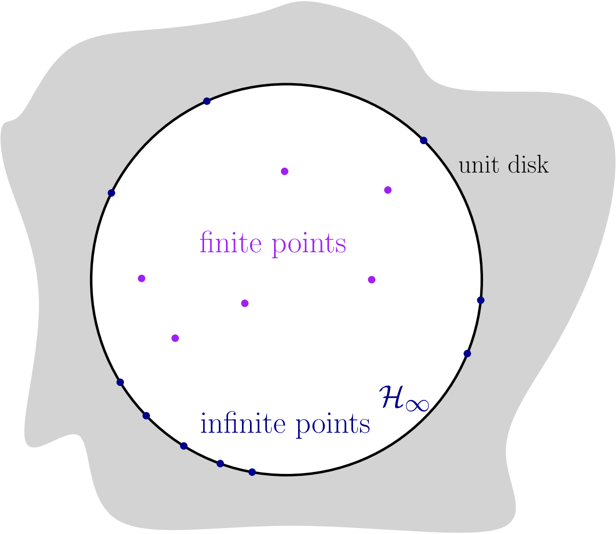
$$\mathcal{G} = \langle a, b, c, d \mid abcd\bar{a}\bar{b}\bar{c}\bar{d} \rangle$$

2-torus

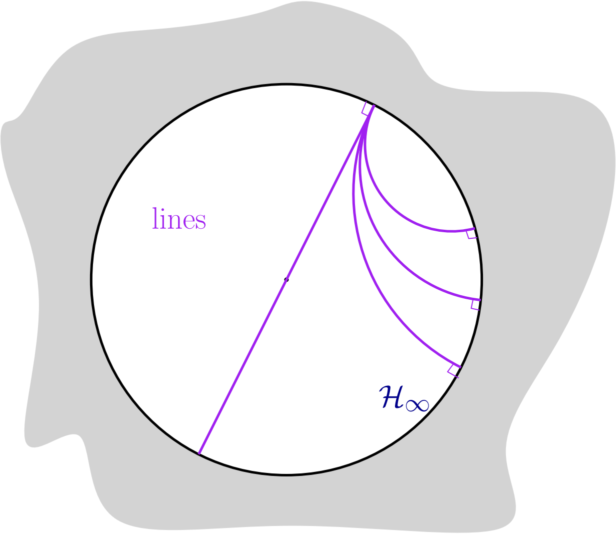
locally hyperbolic metric

Outline

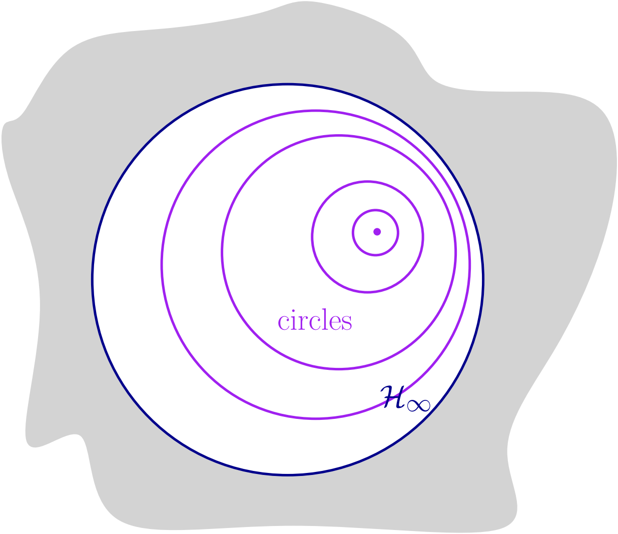
Poincaré disk model of $\mathbb{H}^2$ conformal model



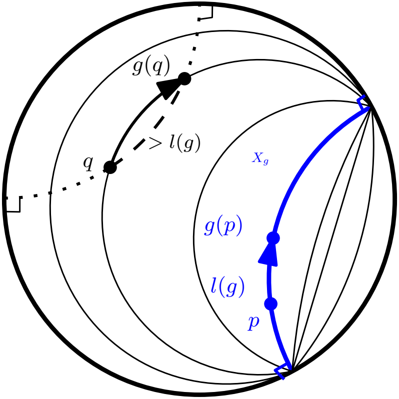
Poincaré disk model of $\mathbb{H}^2$ conformal model



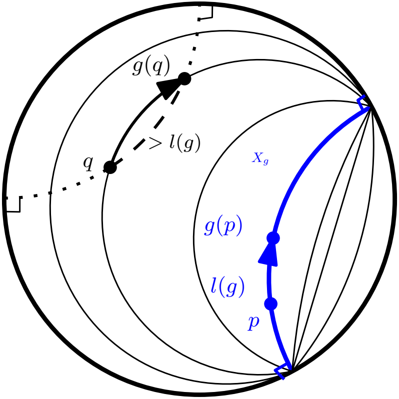
Poincaré disk model of $\mathbb{H}^2$ conformal model



Hyperbolic translations



Hyperbolic translations

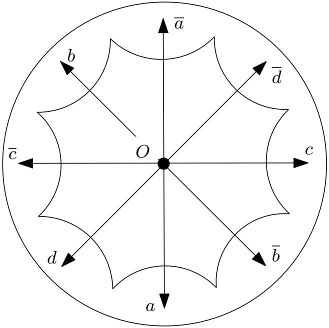


Hyperbolic translations DO NOT COMMUTE

Fuchsian groups

Outline

Definition

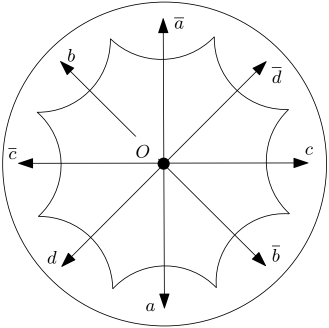


notation: $gO = g, g \in \mathcal{G}$

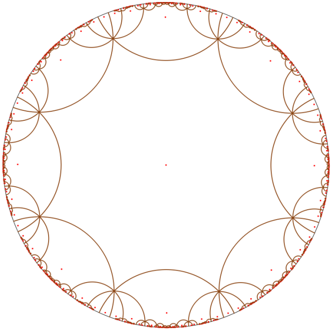
$\mathcal{D}_O(\mathcal{G})$ = region of O

in $Vor(\mathcal{G}O)$

Definition

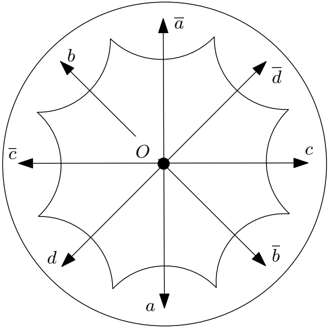


notation: $gO = g, g \in \mathcal{G}$



$Vor(\mathcal{G}O)$

Definition



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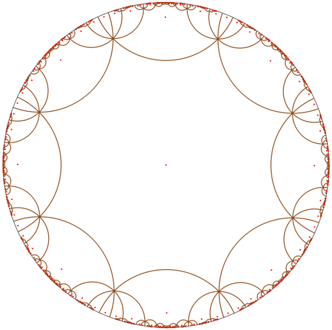


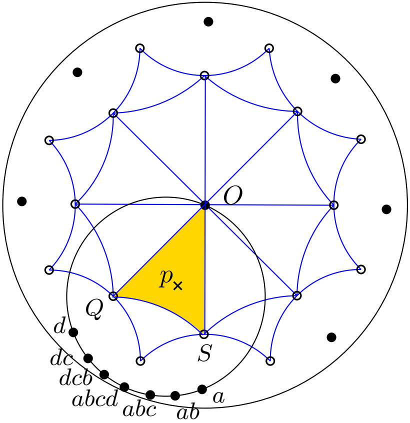
$Vor(\mathcal{G}O)$

$$\mathcal{M} = \mathbb{H}^2/\mathcal{G}$$

$$\mathcal{G} = \langle a, b, c, d \mid abcd\bar{a}\bar{b}\bar{c}\bar{d} \rangle$$

$Vor(\mathcal{G}p), p \neq 0$

[Naatanen] [B.,T.]



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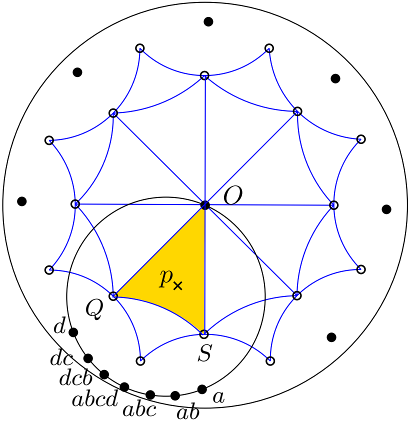
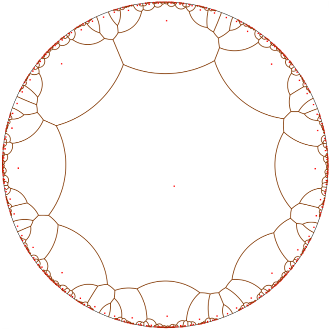


p in interior of OQS



18 sides

$Vor(\mathcal{G}p), p \neq 0$

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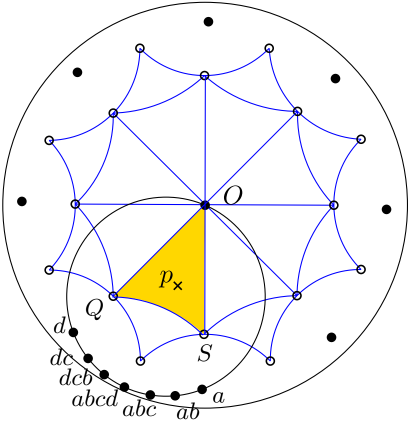

p on $]O, Q[$, $]Q, S[$, or $]S, Q[$

14 sides

$Vor(\mathcal{G}p), p \neq 0$

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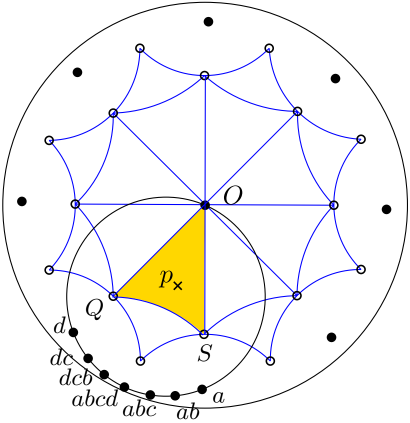
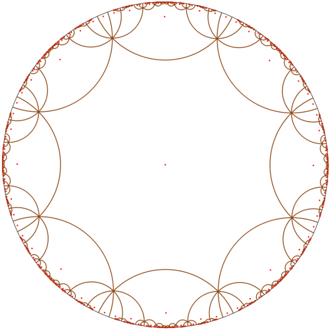


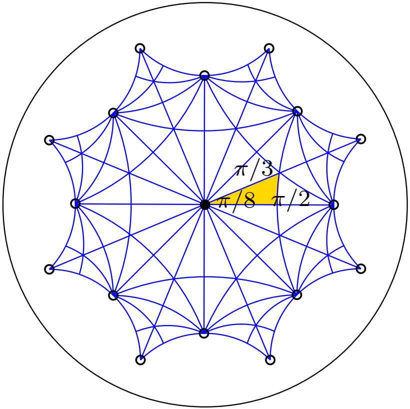
$p \in \{O, Q, S\}$



8 sides

Triangle group $T(2,3,8)$

generated by reflexions in edges of triangle $\pi/2, \pi/3, \pi/8$



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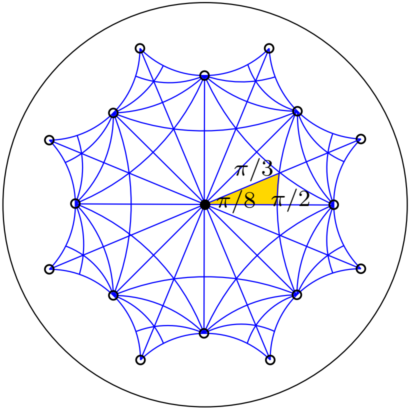

$\mathcal{G}$ normal subgroup
of $T(2,3,8)$
of index 96
without fixed points

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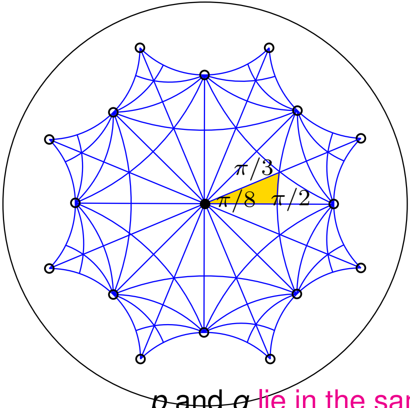


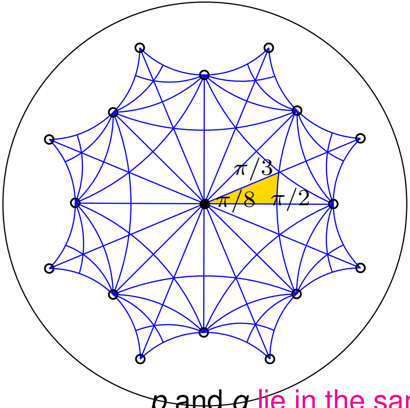
$\mathcal{G}$ normal subgroup of $T(2,3,8)$ of index 96 without fixed points

p and q lie in the same orbit under $T(2,3,8)$

$\Longrightarrow$ the region of p in $Vor(\mathcal{G}p)$ and the region of q in $Vor(\mathcal{G}q)$ are isometric

Triangle group $T(2,3,8)$

generated by reflexions in edges of triangle $\pi/2, \pi/3, \pi/8$



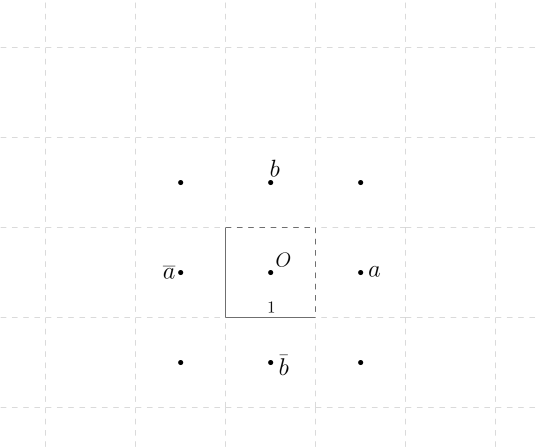
$\mathcal{G}$ normal subgroup
of $T(2,3,8)$
of index 96
without fixed points

p and q lie in the same orbit under $T(2,3,8)$
$\Longrightarrow$ the region of p in $Vor(\mathcal{G}p)$
and the region of q in $Vor(\mathcal{G}q)$
are isometric NOT TRUE OTHERWISE

Outline

2-sheeted covering space of the flat torus

[B. T.] *EuroCG'12*



$$G := \langle a, b \rangle$$

$$\mathbb{T}^2 := \mathbb{R}^2 / G$$

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2-sheeted covering space of the flat torus

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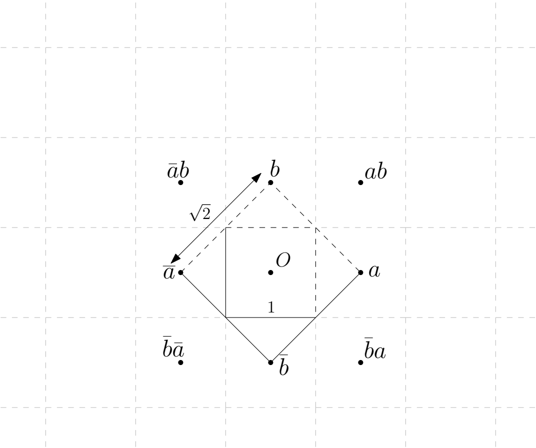

$$G_2 := \langle \bar{a}b, ab \rangle$$

$$\mathbb{T}_2^2 := \mathbb{R}^2 / G_2$$

$$\pi_2 : \mathbb{R}^2 \to \mathbb{T}_2^2$$

2-sheeted covering space of the flat torus

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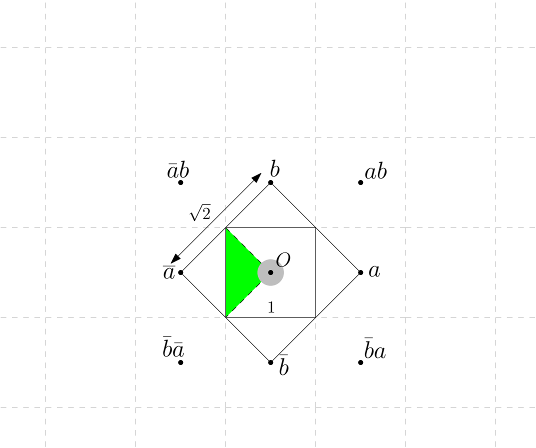

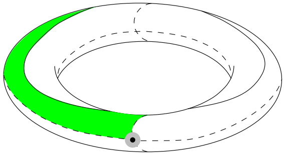

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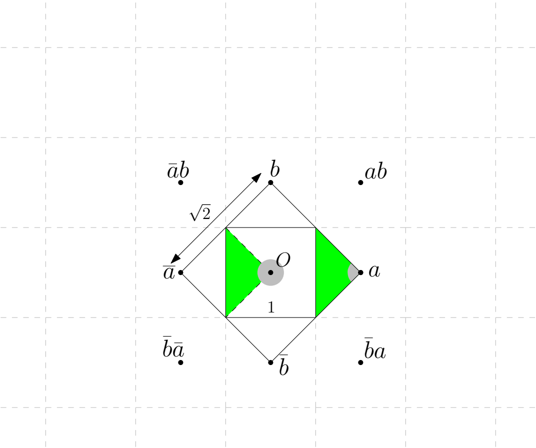
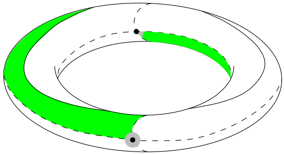
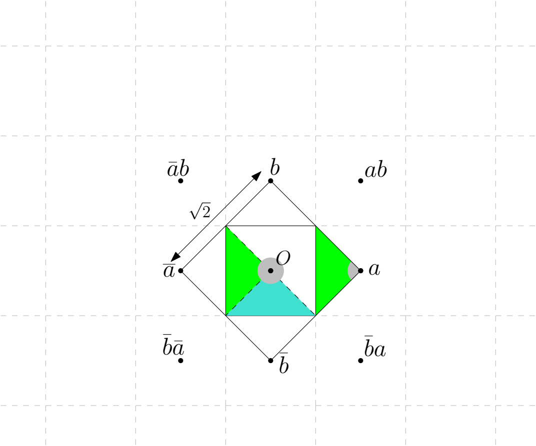
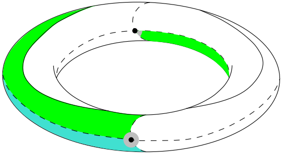




2-sheeted covering space of the flat torus [B. T.] *EuroCG'12*





2-sheeted covering space of the flat torus

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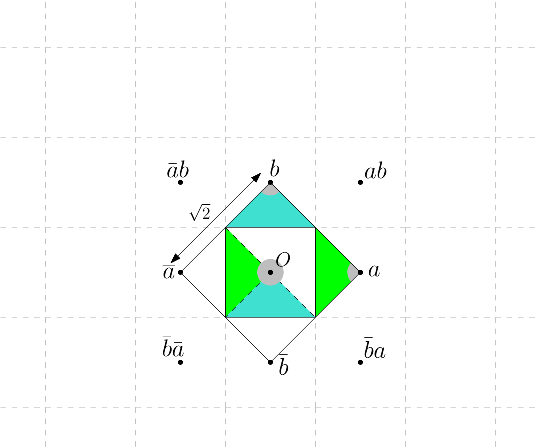
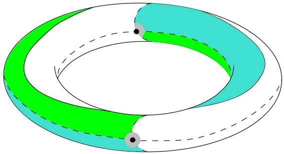
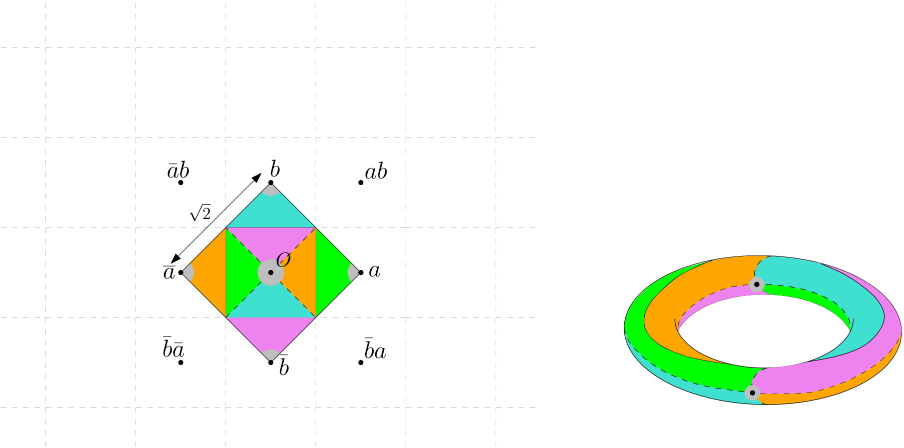




2-sheeted covering space of the flat torus [B. T.] *EuroCG'12*



2-sheeted covering space of the flat torus

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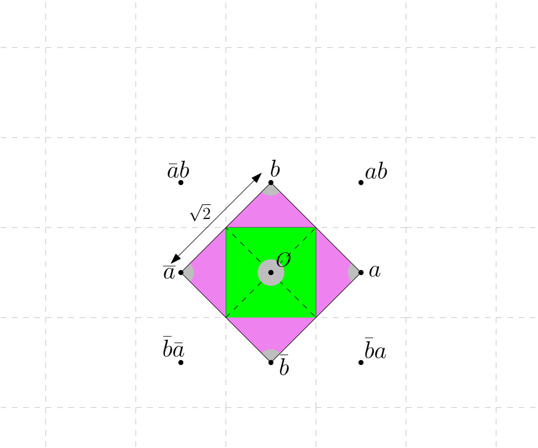
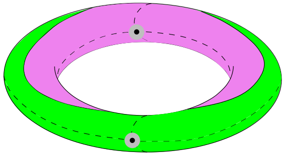
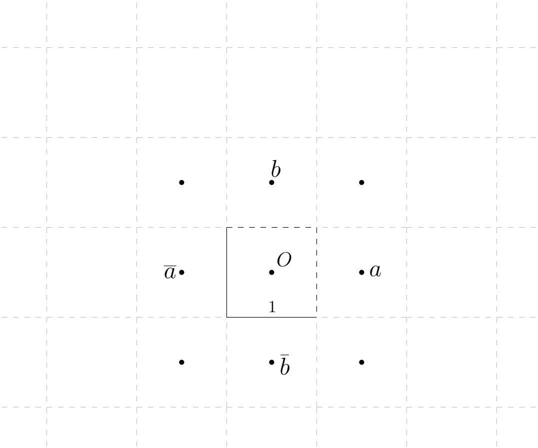




2^k-sheeted covering spaces of the flat torus [B. T.] *EuroCG'12*

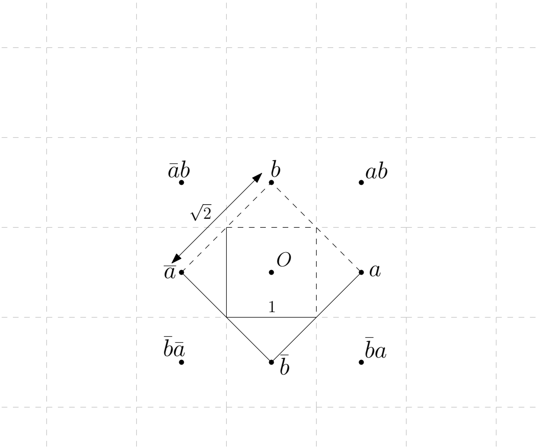
systole of a surface = length of shortest non-contractible loop



$G := \langle a, b \rangle$

$sys(\mathbb{T}^2) = 1$

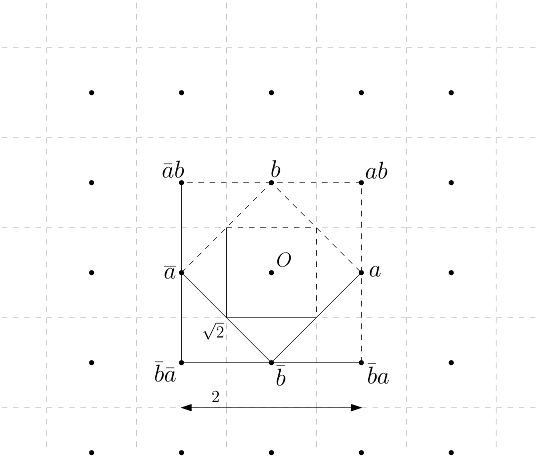
2^k-sheeted covering spaces of the flat torus [B. T.] *EuroCG'12*



$$G_2 := \langle \bar{a}b, ab \rangle$$

$$sys(\mathbb{T}_2^2) = \sqrt{2}$$

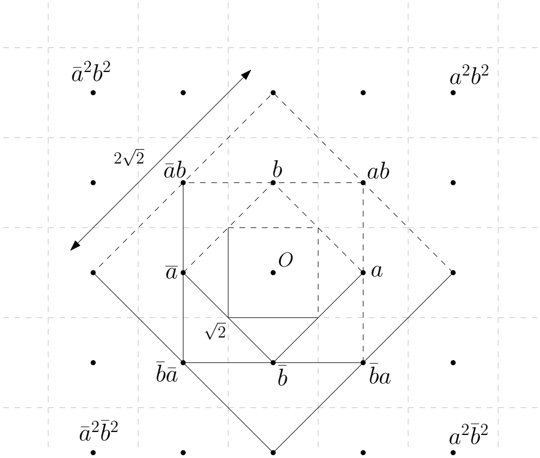
2^k-sheeted covering spaces of the flat torus [B. T.] *EuroCG'12*



$$G_4 := \langle a^2, b^2 \rangle$$

$$sys(\mathbb{T}_4^2) = 2$$

2^k-sheeted covering spaces of the flat torus [B. T.] *EuroCG'12*



$G_8 :=$
$\left\langle \bar{a}^2 b^2, a^2 b^2 \right\rangle$

$sys(\mathbb{T}_8^2) = 2\sqrt{2}$

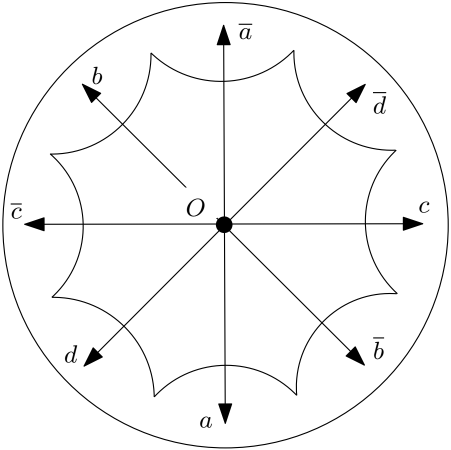
2^k-sheeted covering spaces of the flat torus [B. T.] *EuroCG'12*

$G_{2^{k-1}} = \langle g, h \rangle$

$G_{2^k} = \langle \overline{g}h, gh \rangle$
$= \text{elements of even size in elements of } G_{2^{k-1}}$

$sys(\mathbb{T}^2_k) = \sqrt{2}^k$

Bolza surface $\mathcal{M}$

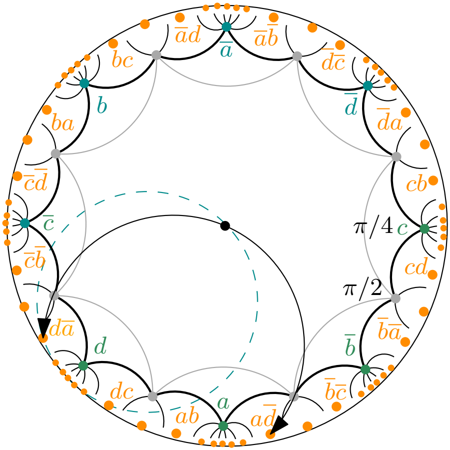


$\mathcal{G} =$
$\langle a, b, c, d \mid abcd\overline{a}\overline{b}\overline{c}\overline{d} \rangle$

$sys(\mathcal{M}) =$
translation_length(a) $\simeq 3.06$

Bolza surface $\mathcal{M}$

$\mathcal{G}_{k+1}$ generated by words of even size in the generators of $\mathcal{G}_k$



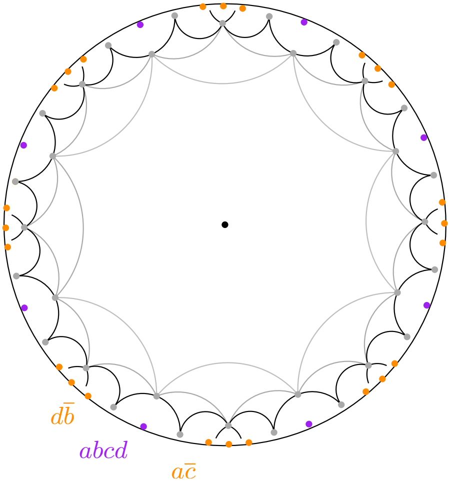
$\mathcal{G}_2 =$
$\langle ab, cd, \bar a\bar b, \bar c\bar d, bc, d\bar a, \bar b\bar c, \bar d a$
$\mid abcd\bar a\bar b\bar c\bar d\rangle$

= only subgroup of index 2 of $\mathcal{G}$ s.t. $\mathcal{D}_O$ invariant by $\rho_{\pi/4}$

$sys(\mathcal{M}_2) = sys(\mathcal{M})$
(fewer shortest loops)

Bolza surface $\mathcal{M}$

$\mathcal{G}_{k+1}$ generated by words of even size in the generators of $\mathcal{G}_k$



$\mathcal{G}_4$ generated by

$d\bar{b}$, $abcd$, $a\bar{c}$

and their conjugates by $\rho_{\pi/4}$

= only subgroup of index 2 of $\mathcal{G}_2$ s.t. $\mathcal{D}_O$ invariant by $\rho_{\pi/4}$

$sys(\mathcal{M}_4) > sys(\mathcal{M})$

Bolza surface $\mathcal{M}$

$\delta_p(\mathcal{G})$ = diameter of largest empty disk in $\mathcal{G}p$
$\delta_{\mathcal{M}} = \sup_p \delta_p(\mathcal{G})$

$$4.89 < \delta_{\mathcal{M}} < 6.62$$

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If $sys(\mathcal{M}_N) > 2\delta_{\mathcal{M}}$ then

$$N > 32$$

($N = 8$ for the flat torus)

$$sys(\mathcal{M}_{128}) > 2\delta_{\mathcal{M}}$$

(using GAP)

Thank you for your attention